Class 14

Theorem 1: f continuous on the circle and $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$, then $S_N f(\theta) \xrightarrow[N\to\infty]{} f(\theta)$ $\forall \theta \in \mathbb{R}$ "uniformly"

Theorem 2: f twice continuously differentiable on the circle. $\Rightarrow S_N f(\theta) \to f(\theta)$ $\forall \theta \in \mathbb{R}$.

Theorem 3: f integrable on the circle $\|S_N f - f\| \xrightarrow[N\to\infty]{} 0$

Theorem 4: $f: \mathbb{R} \to \mathbb{C}$; $f \in C^1(\mathbb{R})$, then $S_N f(\theta) \to f(\theta)$ $\forall \theta \in \mathbb{R}$ uniformly

2π-periodic

Theorem 5: f integrable on the circle, and f differentiable at the point θ_0. Then $S_N(f)(\theta_0) \to f(\theta_0)$

~~Theorem 6: f continuous on the circle, then $S_N f(\theta_0) \to f(\theta_0)$~~

DOES NOT WORK !

SECTION 2.2

PROOF THEOREM 4: Let $M > N \geq 1$

$$|S_M(f)(\theta) - S_N(f)(\theta)| = \left| \sum_{|n|\leq M} \hat{f}(n) e^{in\theta} - \sum_{|n|\leq N} \hat{f}(n) e^{in\theta} \right|$$

$$= \left| \sum_{|n|=N+1}^{M} \hat{f}(n) e^{in\theta} \right| \leq \sum_{|n|=N+1}^{M} |\hat{f}(n)|$$

$$\boxed{\hat{f'}(n) = in\hat{f}(n)}$$

$$|S_M(f)(\theta) - S_N(f)(\theta)| \leq \sum_{|n|=N+1}^{M} \left| \frac{\hat{f'}(n)}{in} \right| = \sum_{|n|=N+1}^{M} \frac{|\hat{f'}(n)|}{|n|} \quad \text{C-S}$$

$$\leq \left(\sum_{|n|=N+1}^{M} |\hat{f'}(n)|^2 \right)^{1/2} \cdot \left(\sum_{|n|=N+1}^{M} \frac{1}{|n|^2} \right)^{1/2}$$

f' integrable on the circle:

$$\|f'\|^2 = \sum_{n\in\mathbb{Z}} |\hat{f'}(n)|^2 \quad \text{(PARSEVAL FOR } f'\text{)}$$

$$|S_M(f)(\theta) - S_N(f)(\theta)| \leq \|f'\| \cdot \left(\sum_{|n|=N+1}^{M} \frac{1}{|n|^2} \right)^{1/2}$$

$$\sum_{|n|=N+1}^{M} \frac{1}{|n|^2} = 2 \sum_{n=N+1}^{M} \frac{1}{n^2}.$$

$$\sum_{n=N+1}^{M} \frac{1}{n^2}$$

$n-1 \leq x \leq n$, $n \geq 2$

$$\frac{1}{n^2} \leq \frac{1}{x^2} \leq \frac{1}{(n-1)^2}$$

$$\int_{n-1}^{n} \frac{1}{n^2}\,dx \leq \int_{n-1}^{n} \frac{dx}{x^2}$$

$$\frac{1}{n^2} \leq \int_{n-1}^{n} \frac{dx}{x^2}$$

$$\frac{1}{n^2} \leq \int_{n-1}^{n} \frac{dx}{x^2} \Rightarrow \sum_{n=N+1}^{M} \frac{1}{n^2} \leq \underbrace{\sum_{n=N+1}^{M} \int_{n-1}^{n} \frac{dx}{x^2}}$$

$$= \int_N^M \frac{dx}{x^2} \leq \int_N^\infty \frac{dx}{x^2} = \frac{x^{-1}}{-1}\Big|_{x=N}^{x=\infty}$$

$$\boxed{\sum_{n=N+1}^{M} \frac{1}{n^2} \leq \frac{1}{N}}$$

$$|S_M(f)(\theta) - S_N(f)(\theta)| \leq \|f'\| \cdot \left(\frac{2}{N}\right)^{1/2}$$

$M > N \geq 2$

$S_M(f)(\theta) \xrightarrow[M\to\infty]{} f(\theta)$

$\Downarrow$ Fix N: $M \to \infty$

$$|f(\theta) - S_N(f)(\theta)| \leq \|f'\| \left(\frac{2}{N}\right)^{1/2}$$

$$\Rightarrow \boxed{|S_N(f)(\theta) - f(\theta)|^2 \leq \|f'\|^2 \cdot \frac{2}{N}}$$

For all $\theta \in \mathbb{R}$ $\quad \forall N \geq 2$

Corollary: $f : \mathbb{R} \to \mathbb{C}$ 2π- periodic ; $C^1(\mathbb{R})$;

$\hat{f}(0) = 0$.

Then: $\|f\| \leq \|f'\|$ (WIRTINGER INEQUALITY)

PROOF: $\widehat{f'}(n) = in\,\hat{f}(n)$

$$\|f'\|^2 = \sum_{n\in\mathbb{Z}} |\widehat{f'}(n)|^2 = \sum_{n\in\mathbb{Z}} |in\hat{f}(n)|^2 = \sum_{n\in\mathbb{Z}} n^2|\hat{f}(n)|^2 = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} n^2|\hat{f}(n)|^2$$

Parseval

$$\|f\|^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \sum_{n\in\mathbb{Z}} 1^2|\hat{f}(n)|^2 = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} |\hat{f}(n)|^2$$

When $n \in \mathbb{Z}, n \neq 0 \Rightarrow n^2 \geq 1$

$$\|f\|^2 = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} 1\cdot|\hat{f}(n)|^2 \leq \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} n^2|\hat{f}(n)|^2 = \|f'\|^2$$

$$\Rightarrow \|f\|^2 \leq \|f'\|^2 \Rightarrow \|f\| \leq \|f'\|$$

THE EQUALITY: $\|f\| = \|f'\|$ HOLDS IF AND ONLY IF $f(x) = \hat{f}(1)e^{ix} + \hat{f}(-1)e^{-ix}$

$$\|f\|^2 = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} |\hat{f}(n)|^2 \leq \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} n^2 |\hat{f}(n)|^2 = \|f'\|^2$$

$a \leq b \leq c$

$a = c$

$\Downarrow$

$a = b = c$

If $\|f\|^2 = \|f'\|^2 \Rightarrow \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} |\hat{f}(n)|^2 = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} n^2 |\hat{f}(n)|^2$

$$\boxed{\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \underbrace{(n^2-1)|\hat{f}(n)|^2}_{\geq 0} = 0}$$

Lemma

$\sum_{n\in\mathbb{Z}} a_n = 0$; $\boxed{a_n \geq 0} \Rightarrow a_n = 0$

BY CONTRADICTION: IF $a_{n_0} \neq 0$

$\Rightarrow 0 = \sum_{n\in\mathbb{Z}} a_n \geq a_{n_0} \Rightarrow a_{n_0} \leq 0$

$0 \leq a_{n_0} \leq 0 \Rightarrow a_{n_0} = 0$

$(n^2-1)|\hat{f}(n)|^2 = 0 \quad \forall n\in\mathbb{Z};\ n\neq 0$

if: $\boxed{n\neq 1, n=-1} \Rightarrow n^2-1\neq 0 \Rightarrow \boxed{\hat{f}(n) = 0}$

$n\in\mathbb{Z}-\{1,1\}$

Therefore:

$$f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx}$$

$$f(x) = \hat{f}(1) e^{ix} + \hat{f}(-1) e^{-ix}$$

$\Updownarrow$

EXTREMIZER

$\|f\| = \|f'\|$

COROLLARY 2: $f: \mathbb{R}\to\mathbb{C}$, 2π-periodic, $f\in C^1(\mathbb{R})$, $\hat{f}(0)=0$,
$g: \mathbb{R}\to\mathbb{C}$, 2π-periodic, $g\in C^1(\mathbb{R})$, Then

$$|\langle f, g\rangle| \leq \|f\|\cdot\|g'\|$$

PROOF: $|\langle f, g\rangle| \underset{C.S.}{\leq} \|f\|\cdot\|g\| \leq \|f\|\cdot\|g'\|$?

$$\langle f, \hat{g}(0)\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{\hat{g}(0)}\,dx = \overline{\hat{g}(0)}\,\frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \overline{\hat{g}(0)}\cdot\hat{f}(0) = 0$$

$$|\langle f;g\rangle| = |\langle f; g-\hat{g}(0)\rangle| \underset{C.S}{\leq} \|f\|\cdot\|g-\hat{g}(0)\|$$

$$|\langle f,g\rangle| = |\langle f, g-\hat{g}(0)\rangle| \le \|f\|.\|\underbrace{g-\hat{g}(0)}_{h}\|$$

corollary for "h"

$$\le \|f\|.\|h\| \le \|f\|.\|h'\|$$

$$\underbrace{h = g-\hat{g}(0)}_{h'=g'} \Rightarrow \hat{h}_{(n)} = \hat{g}_{(n)} - \hat{g}(0) \Rightarrow \hat{h}(0) = 0$$

$$\|f\|.\|g'\|$$

$$\Rightarrow |\langle f,g\rangle| \le \|f\|.\|g'\|$$

PROOF OF THEOREM 5: Let θ_0 where f is diff.

$f: [-\pi,\pi] \to \mathbb{C}$. D_N: DIRICHLET KERNEL $\to \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(\theta)d\theta = 1$

$$(S_N f)(\theta_0) - f(\theta_0) = (f * D_N)(\theta_0) - f(\theta_0)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t)\, f(\theta_0 - t)\,dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(t)\, f(\theta_0)\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t).\,(f(\theta_0-t) - f(\theta_0))\,dt$$

$$D_N(t) = \frac{\sin((N+\frac{1}{2})t)}{\sin(t/2)} \qquad = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t).\,t.\;F(t) \quad dt$$

$$\boxed{F(t).t = f(\theta_0 - t) - f(\theta_0)}$$

$$F(t) = \begin{cases} \dfrac{f(\theta_0 - t) - f(\theta_0)}{t} & ;\ t \ne 0 \\ -f'(\theta_0) & ;\ t = 0 \end{cases}$$

$$\lim_{t\to 0} F(t) = \lim_{t\to 0} \frac{f(\theta_0 - t) - f(\theta_0)}{t} = -f'(\theta_0) = F(0)$$

$$D_N(t).\,(f(\theta_0 - t) - f(\theta_0)) = (D_N(t).\,t)\,F(t) \qquad \forall t$$

$$(S_N f)(\theta_0) - f(\theta_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(D_N(t) \cdot t\right) \cdot F(t)\, dt \quad \ldots (*)$$

$$D_N(t) = \sum_{k=-N}^{N} e^{ikt} = \frac{\sin\left(\left(N+\frac{1}{2}\right)t\right)}{\sin\left(\frac{t}{2}\right)}$$

exists

$$\lim_{t\to 0} \frac{t}{\sin(t/2)} = 2$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

In (*):

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\sin\left(\left(N+\tfrac{1}{2}\right)t\right)}_{\sin\left(Nt+\frac{t}{2}\right)} \cdot \frac{t}{\sin(t/2)}\, F(t)\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sin Nt \cdot \cos\frac{t}{2} + \cos Nt \cdot \sin\frac{t}{2}\right) \frac{t}{\sin(t/2)}\, F(t)\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{t\, F(t)\, \cos(t/2)}{\sin(t/2)} \cdot \sin Nt + \frac{1}{2\pi}\int_{-\pi}^{\pi} t\, F(t) \cdot \cos Nt\, dt$$

$tF(t)$ is 2π-periodic

$\frac{\cos(t/2)}{\sin(t/2)}$ is 2π-periodic

$\frac{t\,F(t)\cos(t/2)}{\sin(t/2)}$ is 2π-periodic

RIEMANN LEBESGUE LEMMA:

f integrable $\Rightarrow$ $\hat{f}(n) \to 0$

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx = 0$$

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\cos(nx) - i\sin(nx)\right) dx = 0$$

$$\lim_{n\to\infty} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx - i \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\, dx\right) = 0$$

$$\begin{cases} \lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx = 0 \\ \lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\, dx = 0 \end{cases}$$

$f = u + iv \Rightarrow \hat{f} = \hat{u} + i\,\hat{v}$

$\hat{u}(n) \to 0$

$\hat{v}(n) \to 0$

$f = u + iv \Rightarrow \hat{f} = \hat{u} + i\hat{v}$

Using R.L. lemma: f, u, v integrable on the circle

Then $\hat{u}(n) \to 0$, $\hat{v}(n) \to 0$

$\hat{f}(n) \to 0$ $\quad u: R \to R$

$$\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)\, e^{-inx}\, dx \to 0}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(x) \cos(nx)\, dx \to 0$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(x) \sin(nx)\, dx \to 0$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} v(x) \cos(nx)\, dx \to 0$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} v(x) \sin(nx)\, dx \to 0$$

$$\boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cos(nx)\, dx}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x) \cos(nx)\, dx \to 0$$

$$+ \frac{1}{2\pi}\int_{-\pi}^{\pi} i\, v(x) \cos(nx)\, dx \to 0$$

RIEMANN LEBESGUE LEMMA

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \to \underline{0}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \to \underline{0}$$

CONCLUSION: When $N \to \infty$

$$\underline{S_N(f)(\theta_0) - f(\theta_0) \to 0}$$